Waves & Acoustics

Lecture-II

For

2nd Semester (Hons.)

Paper-CC4

Department of Physics

SMHGGDCW

Superposition of Two Perpendicular Harmonic Oscillations

So far we have seen the superposition of waves which are oscillating in the same direction with different phase, different amplitude and different frequencies.

The resultant wave due to these superposition have been discussed earlier already.

In this lecture we will see what will happen when two SHMs oscillating in perpendicular direction combine each other.

It will be easier if we see the pictorial diagrams of formation of the resultant wave. We will also look at the mathematical formulation for the same.

OSCILLATIONS WITH EQUAL FREQUENCIES

ANALITICAL METHOD:

Two perpendicular harmonic oscillations can be represented as-

$$x = A_1 Cos(\omega t) \qquad \& \qquad y = A_2 Cos(\omega t + \phi) \qquad ....1$$

We can find the locus of the instantaneous position of the particle by eliminating the time from the above equations.

Therefore by rearranging above equation we can write-

$$\frac{x}{A_1} = Cos(\omega t) \qquad \& \qquad \frac{y}{A_2} = Cos(\omega t + \phi) \qquad ....2$$

Expanding the Cosine term in the 2nd equation we have

$$\frac{y}{A_2} = \mathrm{Cos}(\omega t)\mathrm{Cos}(\phi) - \mathrm{Sin}(\omega t)\mathrm{Sin}(\phi)$$

$$\Rightarrow \quad \frac{y}{A_2} = \frac{x}{A_1}\mathrm{Cos}(\phi) - \left(1 - \frac{x}{A_1}\right)^{\frac{1}{2}} \mathrm{Sin}(\phi)$$

$$\Rightarrow \quad \frac{y}{A_2} - \frac{x}{A_1}\mathrm{Cos}(\phi) = \left(1 - \frac{x}{A_1}\right)^{\frac{1}{2}} \mathrm{Sin}(\phi)$$

$$\Rightarrow \quad \left(\frac{y}{A_2} - \frac{x}{A_1}\mathrm{Cos}(\phi)\right)^2 = \left(1 - \frac{x}{A_1}\right)\mathrm{Sin}^2(\phi)$$

$$\Rightarrow \quad \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - 2\frac{xy}{A_1 A_2}\mathit{Cos}(\phi) = \mathrm{Sin}^2(\phi) \qquad \ldots .3$$

Nature of The Resultant Motion

Previous equation is the general equation of an ellipse whose axes are inclined to the coordinate axes. Hence the resultant motion of two perpendicular SHMs of equal frequencies is in general an ellipse.

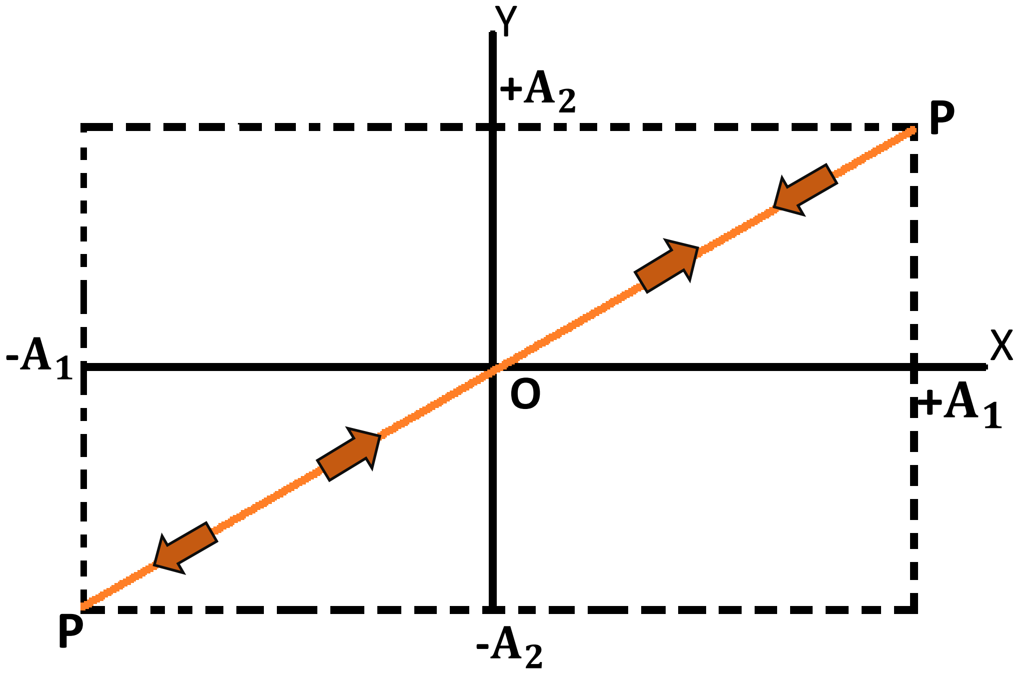
Let us now discuss some special cases for some specific values of $\boldsymbol{\phi}$

Case-1

$\boldsymbol{\phi = 0}$

For $\boldsymbol{\phi = 0}$ the equation (3) becomes

$$\left(\frac{\mathbf{y}}{\mathbf{A_2}} - \frac{\mathbf{x}}{\mathbf{A_1}}\right)^2 = 0 \quad \Rightarrow \quad \left(\mathbf{y} - \frac{\mathbf{A_2}}{\mathbf{A_1}}x\right)^2 = 0$$



Equation (4) represents a pair of coincident straight lines passing through the origin with a slope of $\frac{A_2}{A_1}$ as shown in the figure.

Case-2

For $\phi = \frac{\pi}{2}$ the equation (3) becomes

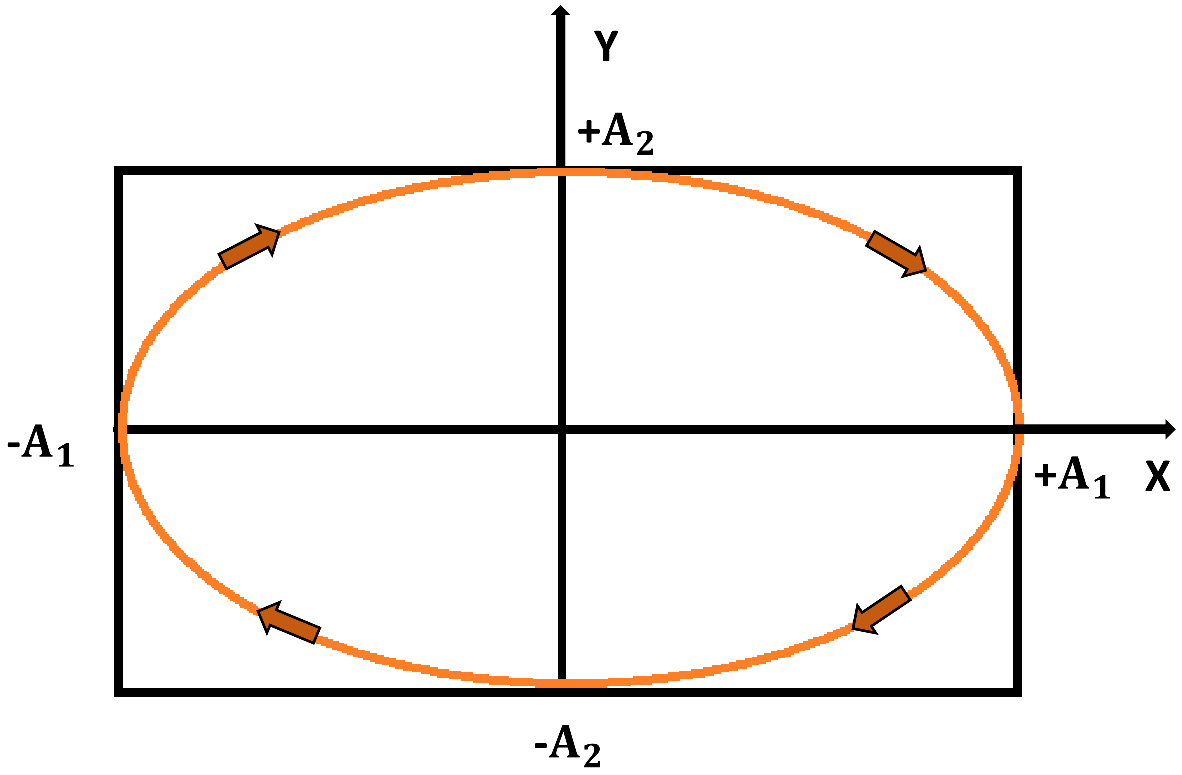
$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1 \qquad \ldots . 5$$

The above equation represents an ellipse whose principal axes lie along x-axis and y-axis.

For $\phi = \frac{\pi}{2}$ the two SHMs becomes-

$$x = A_1 Cos(\omega t) \quad \& \quad y = A_2 Cos(\omega t + \pi/2)$$

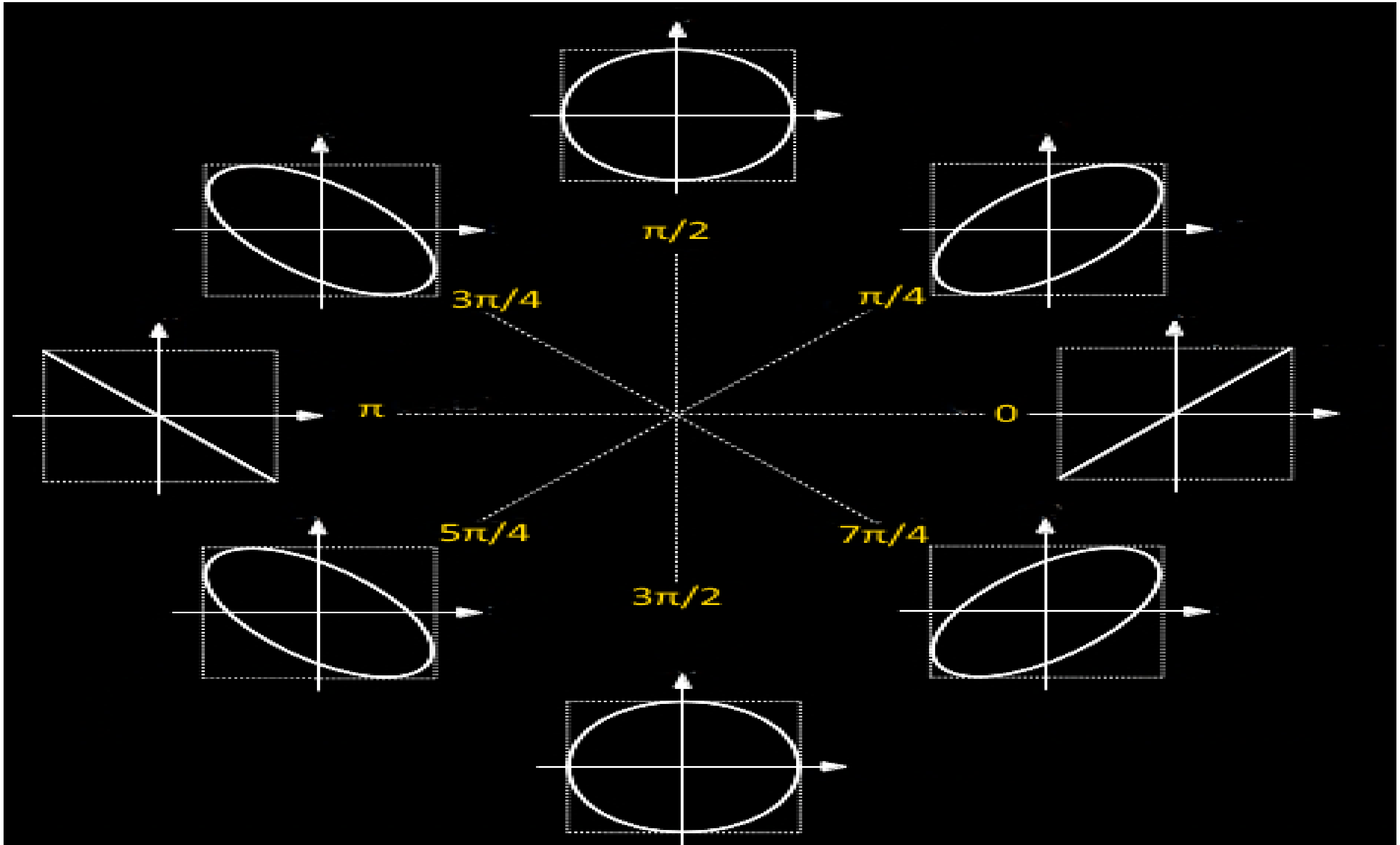
$$\Rightarrow x = A_1 Cos(\omega t) \quad \& \quad y = -A_2 Sin(\omega t)$$



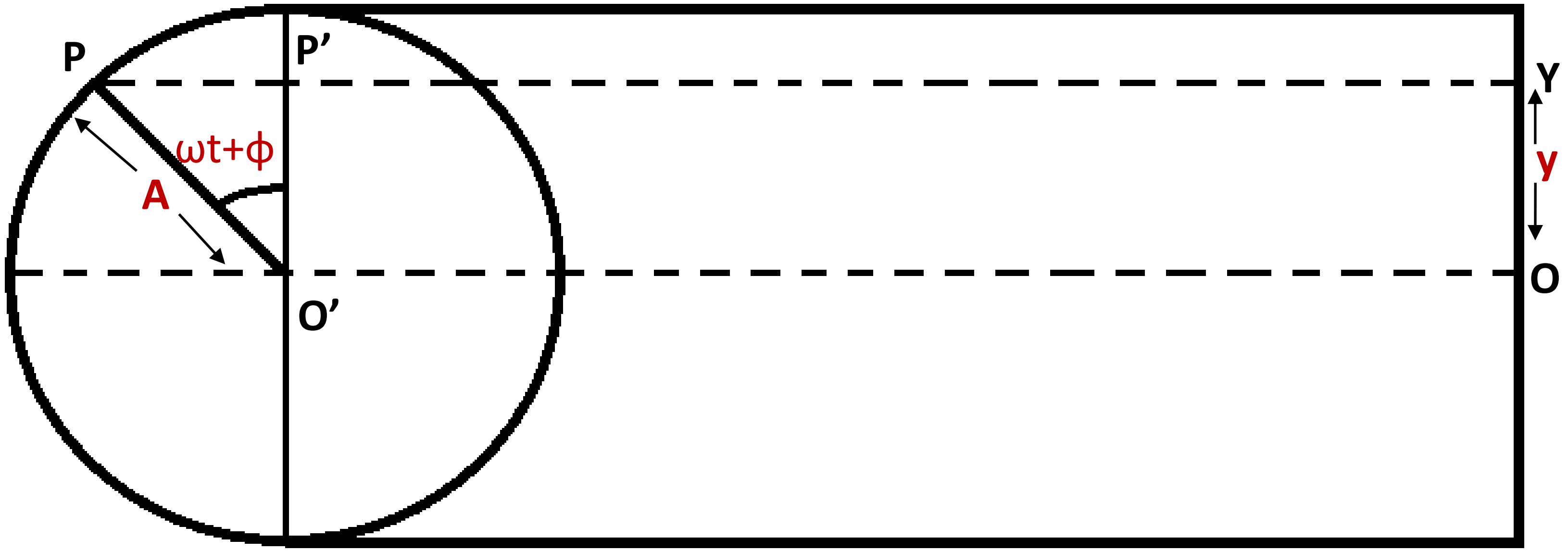
Therefore we can see the ellipse will be a right circular in nature.

For $A_1 = A_2$ this will become a circle.

The Sequence of motion for a few values of ϕ ranges from 0 to 2π are shown below.



Representation of SHM by Rotating Vector method:



We first consider a circle of radius **A** having centre at **O. OP** represents the position of the rotating vector at any instant of time t. ω is the angular velocity of the rotation. This circle represents a SHM along y axis having the amplitude **A** and angular frequency **ω.**

The projection of **OP** on the y-axis **(O'P'=OY=y)** represents the y displacement of the particle constituting the SHM.

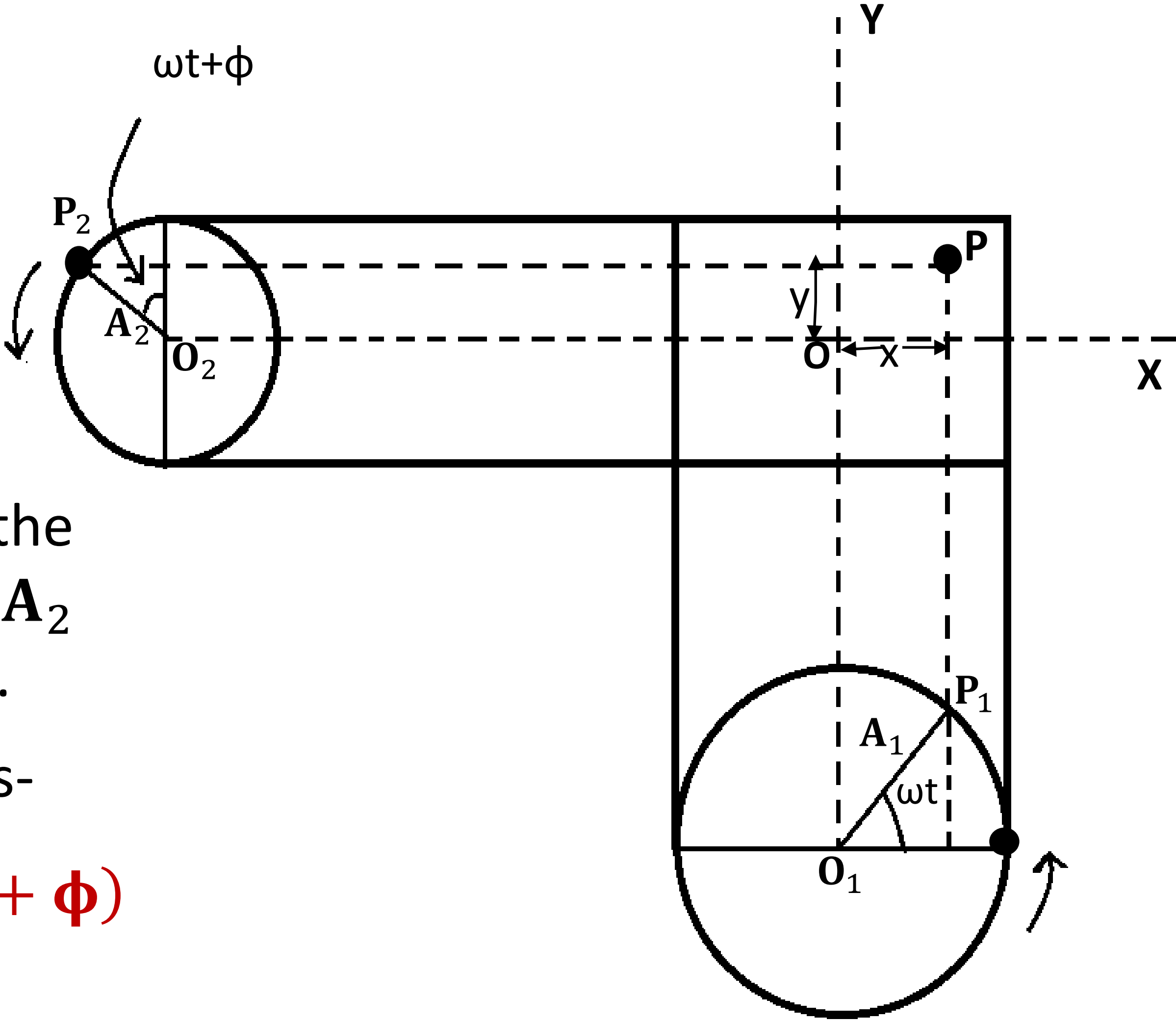
From the geometry we an write-

$$y = ACos(\omega t + \phi)$$

Which represents the equation of the SHM along y-axis.

Superposition of Two Perpendicular Harmonic Oscillations With equal frequencies

GRAPHICAL METHOD:



From the fig. we can see that the two perpendicular SHM is represented as two circle by using the rotating vector method as we have seen earlier.

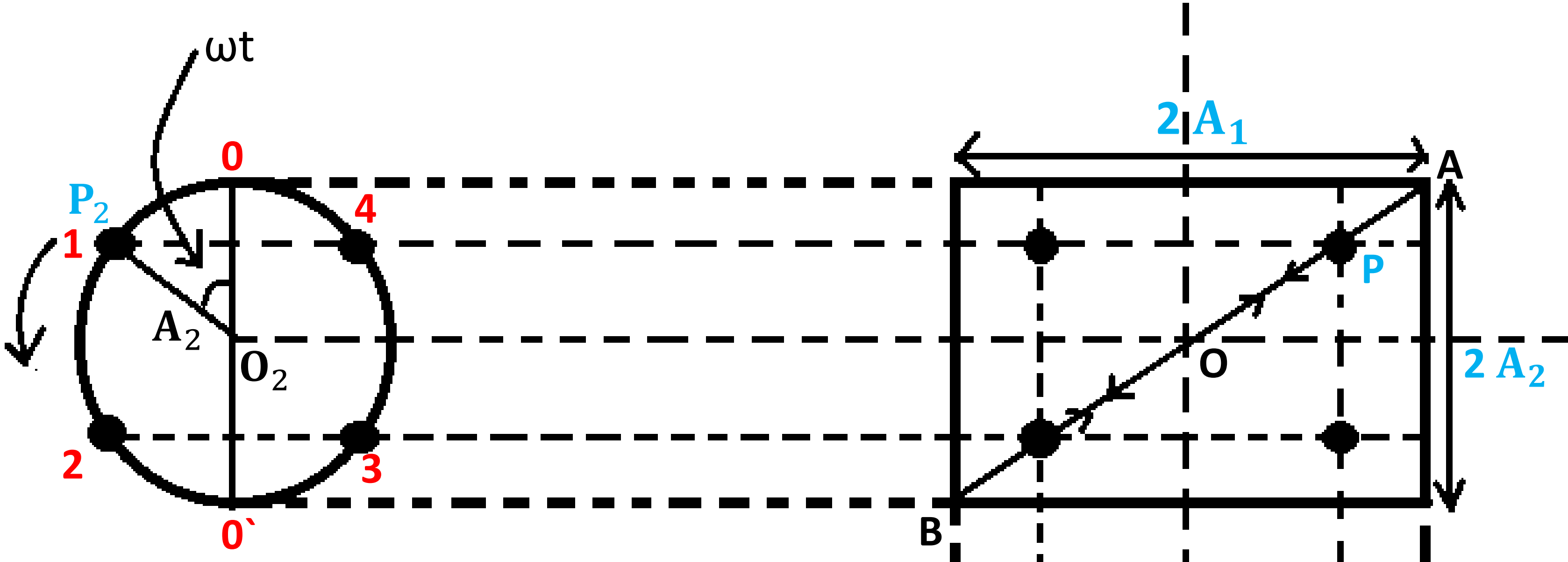
The circle of radius $\mathbf{A_1}$ centred at $\mathbf{O_1}$ represents the oscillation along x-axis where The circle of radius $\mathbf{A_2}$ centred at $\mathbf{O_2}$ represents the oscillation along y-axis.

From the geometry it easy to write the equations as-

$\mathbf{x = A_1 Cos(\omega t)}$ **&** $\mathbf{y = A_2 Cos(\omega t + \phi)}$

The graphical representation is known as Lissajous figure.

Now we will discuss about some special cases for some specific values of ϕ .

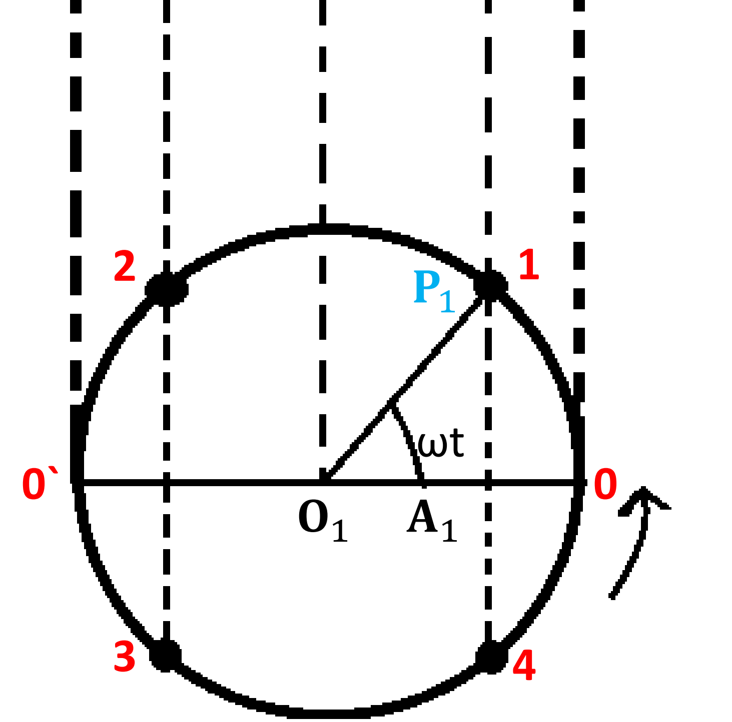


Case-1

$\phi=0$

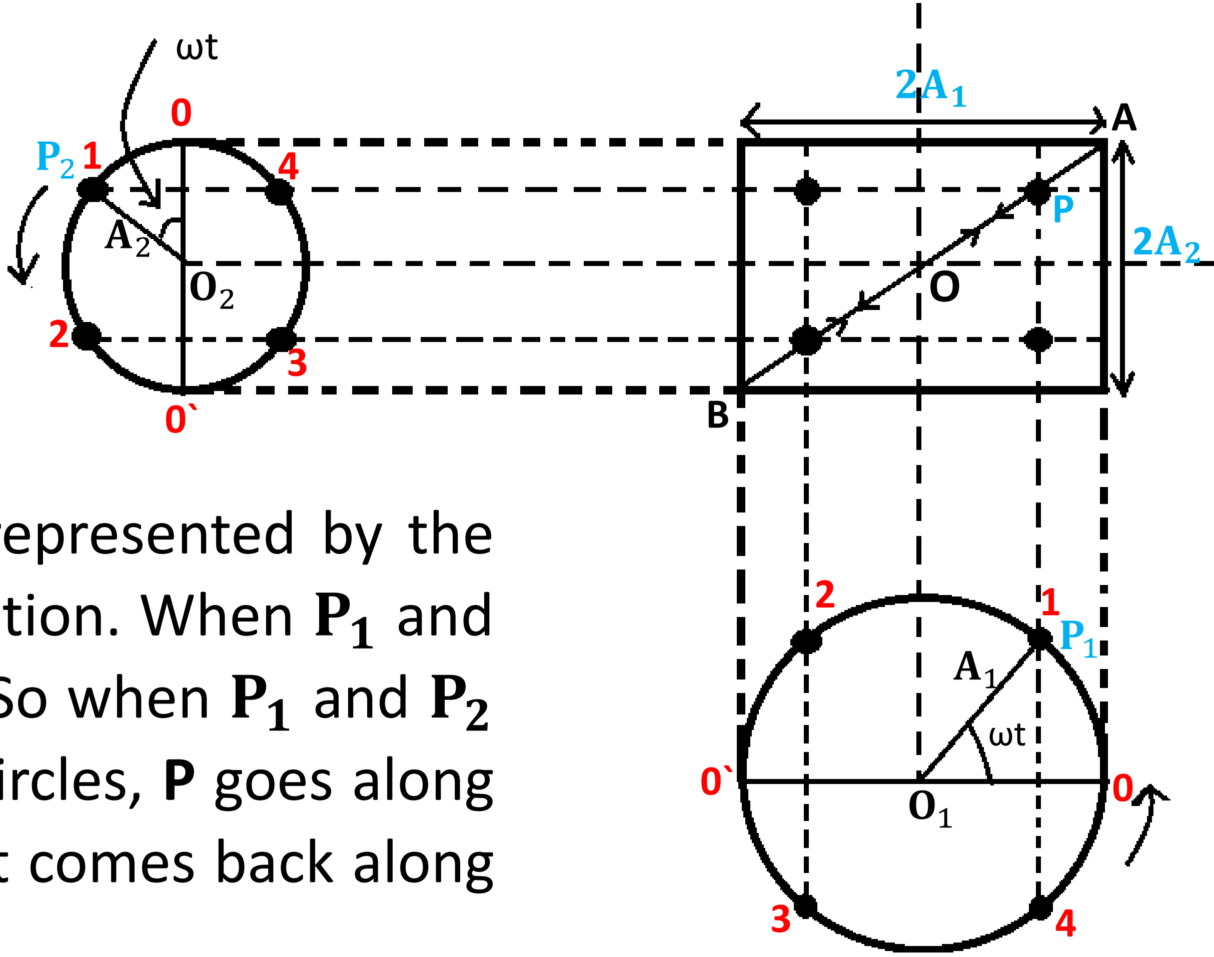
For this case the perpendicular motions are

$$\mathbf{x = A_1 Cos(\omega t)} \qquad \& \qquad \mathbf{y = A_2 Cos(\omega \mathit{t})}$$



Since the phase difference is zero both the particles $\mathbf{P_1}$ and $\mathbf{P_2}$ starts at the same time from the point marked as 0 on each circle. And the resultant position of is represented by the point **A** on the rectangular section**.** As $\mathbf{P_1}$ and $\mathbf{P_2}$ moves along the circle with same angular velocity **P** traces down the straight

Line passing through the centre of the rectangle. On each circle four positions of $\mathbf{P_1}$ and $\mathbf{P_2}$ have been shown namely 1,2,3,4, which are shown as black dots on each circle.

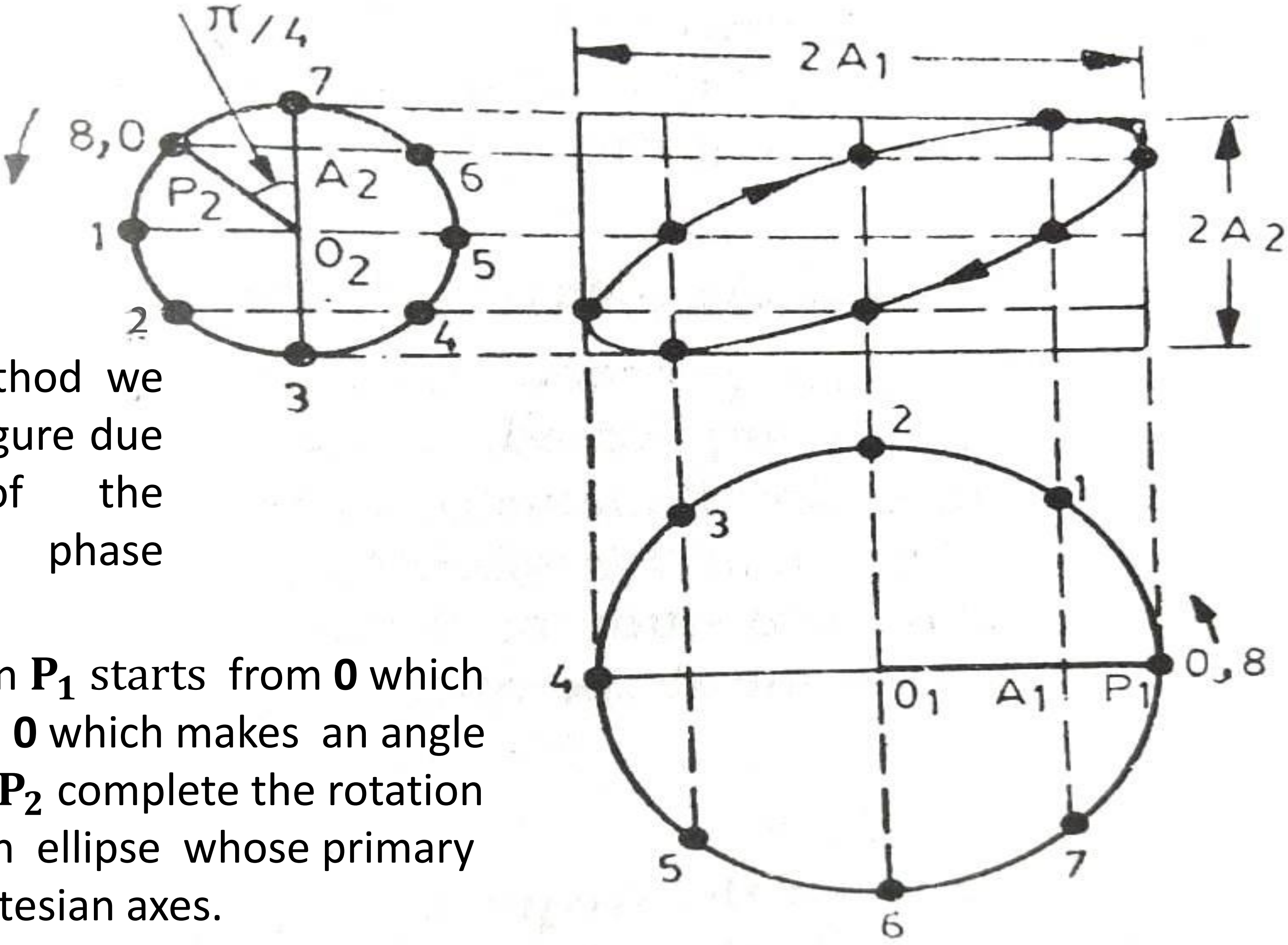


And the corresponding position of **P** has been represented by the black dots(crossing points) on the rectangular section. When $\mathbf{P_1}$ and $\mathbf{P_2}$ reaches the point **0`**, **P** comes to the point **B**. So when $\mathbf{P_1}$ and $\mathbf{P_2}$ complete their rotation around their respective circles, **P** goes along the straight starting from **A** to **B** and then again it comes back along the same path to **A** as shown in the fig.

So the resultant oscillation of two perpendicular SHM with same frequency and zero phase difference will be a pair of straight lines as we have already seen in equation (4) in the previous lecture.

Case-2

$\phi=\pi/4$



Similar to the previous method we can determine the Lissajous figure due to the superposition of the Perpendicular waves having phase difference of $\pi/4$.

In this case it is clear that when $\mathbf{P_1}$ starts from **0** which is on the x- axis, $\mathbf{P_2}$ starts from **0** which makes an angle $\pi/4$ with the y-axis. As $\mathbf{P_1}$ and $\mathbf{P_2}$ complete the rotation the resultant path becomes an ellipse whose primary Axes are not parallel to the Cartesian axes.

For $\phi=\pi/2$ the principle axes of the ellipse will be parallel to the Cartesian axes.

Superposition of Two Perpendicular Harmonic Oscillations With unequal frequencies

Till now we have seen the superposition of waves having equal frequencies.

In this section we will discuss about the superposition of two perpendicular waves having unequal frequencies. First we will find the shape of Lissajous fig. by analytical calculation and then we will plot the Lissajous figure by rotating vector method graphically. Let us take perpendicular waves having frequency ration 1:2. The equation of the waves are

$$\mathbf{x = A_1 Cos(\omega t)} \quad \& \quad \mathbf{y = A_2 Cos(2\omega t + \phi)} \quad \ldots.1$$

In analytical method, we find the locus of the instantaneous particle position by eliminating time t from the above equations. Expanding the 2nd equation we get-

$$\frac{y}{A_2} = \mathrm{Cos}(2\omega t)\cos(\phi) - \mathrm{Sin}(2\omega t)\mathrm{Sin}(\phi)$$

$$\Rightarrow \frac{y}{A_2} = (2\mathrm{Cos}^2\omega t - 1)\cos(\phi) - 2\mathrm{Sin}(\omega t)\mathrm{Cos}(\omega t)\mathrm{Sin}(\phi)) \quad \ldots.2$$

Now $\mathrm{Cos}(\omega t) = \frac{x}{A_1}$ & $\mathrm{Sin}(\omega t) = \sqrt{1 - \frac{x^2}{A_1^2}}$

Therefore from equation (2) we can write

$$\frac{y}{A_2} = \left(2\frac{x^2}{A_1^2} - 1\right)\cos(\phi) - 2\frac{x}{A_1}\sqrt{1 - \frac{x^2}{A_1^2}}\,\mathrm{Sin}(\phi)$$

➡ $$\left(\frac{y}{A_2} + \cos\phi\right) - 2\frac{x^2}{A_1^2}\mathrm{Cos}(\phi) = -2\frac{x}{A_1}\sqrt{1 - \frac{x^2}{A_1^2}}\,\mathrm{Sin}(\phi)$$

➡ $$\boxed{\left(\frac{y}{A_2} + \cos\phi\right)^2 + \frac{4x^2}{A_1^2}\left(\frac{x^2}{A_1^2} - 1 - \frac{y}{A_2}\mathrm{Cos}(\phi)\right) = 0} \qquad \ldots.3$$

This is an equation of 4th degree which, in general, represents a closed curve having two loops.

Now we will consider some special cases corresponding to some specific values of ϕ

Case-1

ϕ= 0

For $\phi = 0$ the equation(3) becomes

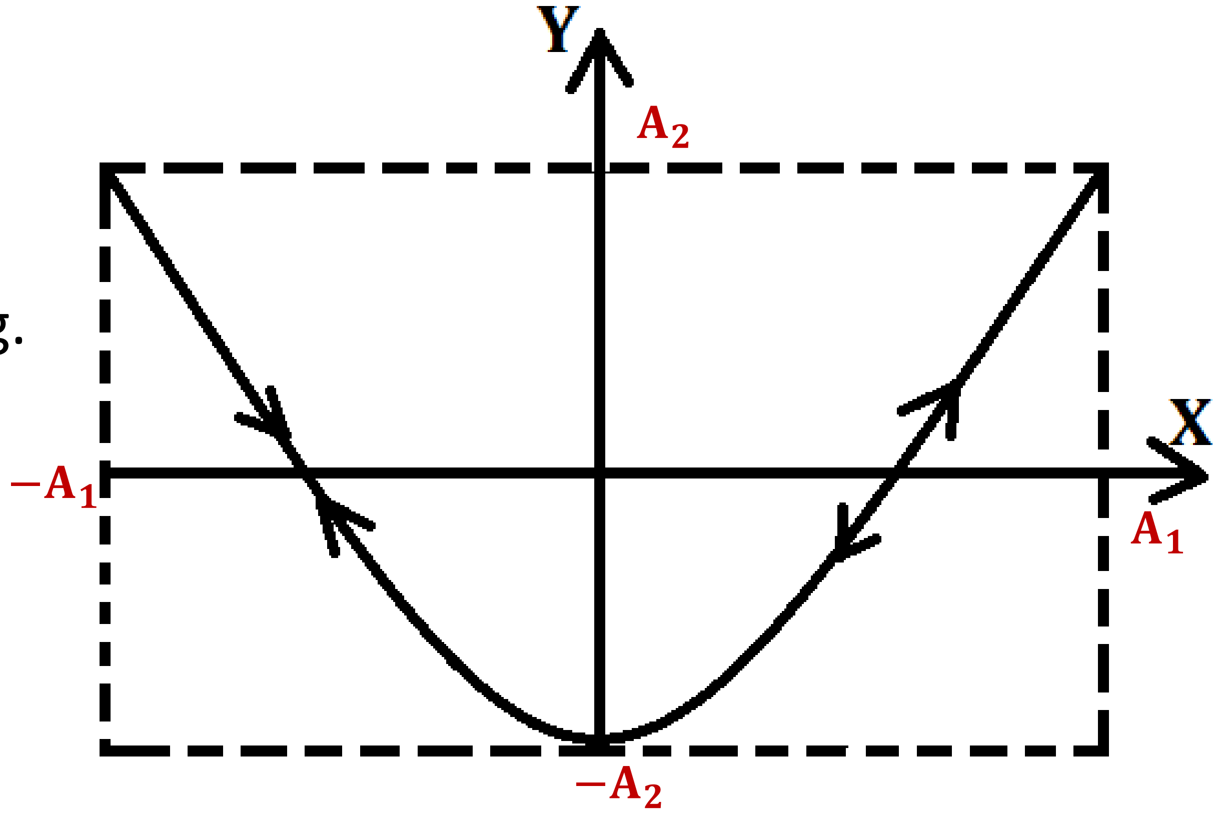
$$\left(\frac{y}{A_2}+1\right)^2+\frac{4x^2}{A_1^2}\left(\frac{x^2}{A_1^2}-1-\frac{y}{A_2}\right)=0$$

$$\Rightarrow \quad \left(\frac{y}{A_2}+1-\frac{2x^2}{A_1^2}\right)^2=0$$

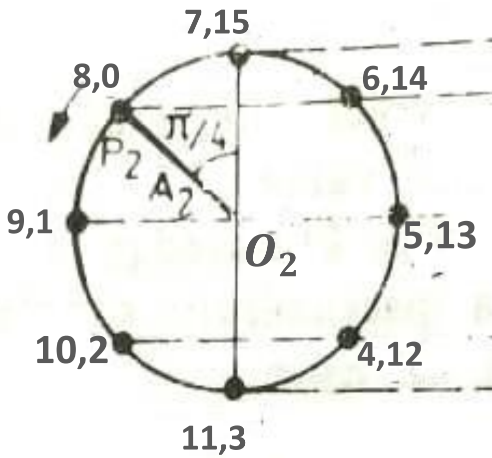
The above equation represents a pair of parabolas With their vertices at $(\mathbf{0},-\mathbf{A_2})$ as shown in the fig. aside.

The equation of each parabola becomes-

$$\frac{y}{A_2}+1-\frac{2x^2}{A_1^2}=0 \quad \Rightarrow \quad \boxed{x^2=\frac{A_1^2}{2A_2}(y+A_2)}$$

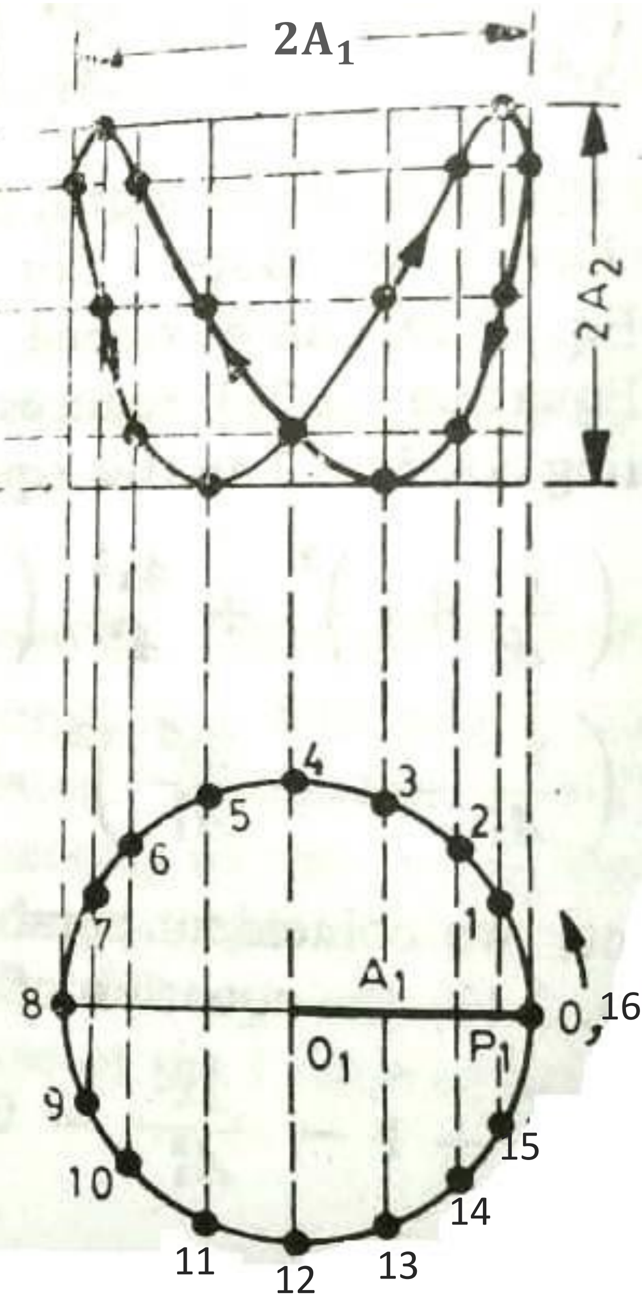


For other values of ϕ equation (3) becomes very Cumbersome. In those cases we can plot the Lissajous fig. by graphical method quite conveniently.



The fig. aside shows the Lissajous fig using rotating vector method for

ϕ=π/4 and $\boldsymbol{\omega_2 = 2\omega_1}$



The rotating vector $\mathbf{O_2P_2}$ makes an angle $\boldsymbol{\pi/4}$ at t=0 with the y-axis but the rotating vector $\mathbf{O_1P_1}$ is along the x-axis at that instant. The y oscillation is twice as fast as x oscillation and that's why we divide the circle of radius $\mathbf{A_2}$ into 8 equal parts and circle of radius $\mathbf{A_1}$ into 16 equal parts. Thus during one complete cycle of $\boldsymbol{\omega_2}$ one goes through only half cycle of $\boldsymbol{\omega_1}$. And the resultant motion will consist of two close loops as shown in the fig.

For the frequencies with ratio 1:n there will be n number of loops in the Lissajous fig.

Few cases for oscillation with different frequency ratios the Lissajous fig is shown below for $\phi=0$